MUMBAI UNIVERSITY

SEMESTER – 2

APPLIED MATHEMATICS SOLVED PAPER – DEC 18

N.B:- (1) Question no.1 is compulsory.

(2) Attempt any 3 questions from remaining five questions.

Q.1 a) Evaluate $\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}} dx$. **[3]**

ANS: $I = \int_0^\infty \frac{e^{-x^3}}{\sqrt{x}} dx.$

Put $x^3 = t$

$\therefore x = t^{\frac{1}{3}}$

$dx = \frac{1}{3} t^{\frac{-2}{3}}$

$\therefore I = \int_0^\infty e^{-t}.t^{-\frac{1}{6}}.\frac{1}{3}.t^{\frac{-2}{3}}\, dt$

$\therefore I = \int_0^\infty e^{-t}.t^{-\frac{5}{6}}\, dt$

$\therefore \boldsymbol{I = \frac{1}{3}\left|\overline{\frac{1}{6}}\right.}$**.**

b) Find the length of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$ from y = 1 to y = 2. [3]

ANS: We have $x = \frac{y^3}{3} + \frac{1}{4y}$

Diff w.r.t. y, we get

$\frac{dx}{dy} = y^2 - \frac{1}{4y^2}$

$$1 + (\frac{dx}{dy})^2 = 1 + \left(y^2 - \frac{1}{4y^2}\right)^2 = y^4 + \frac{1}{2} + \frac{1}{16y^4} = \left(y^2 + \frac{1}{4y^2}\right)^2$$

We know that,

$$s = \int_1^2 \sqrt{1 + (\frac{dx}{dy})^2}\, dy$$

$$s = \int_1^2 \sqrt{\left(y^2 + \frac{1}{4y^2}\right)^2}\, dy$$

$$s = \left[\frac{y^3}{3} - \frac{1}{4y}\right]_1^2$$

$$s = \frac{8}{3} - \frac{1}{8} - (\frac{1}{3} - \frac{1}{4})$$

$$\boldsymbol{s = \frac{59}{24}}$$

c) Solve $(\boldsymbol{D^2 + D})\boldsymbol{y} = \boldsymbol{e^{4x}}$. **[3]**

ANS: For auxiliary equation,

$D^2 + D$= 0

Solving we get,

D = -1, 0.

∴C.F. = $C_1 e^{-x} + C_2 e^{0x}$

$\therefore C.F = C_1 e^{-x} + C_2$

For P.I.,

$$y = \frac{e^{4x}}{D^2+D}$$

Now, put D = 4

$$\therefore y = \frac{e^{4x}}{4^2+4} = \frac{e^{4x}}{20}$$

∴ The complete solution is,

$$\boldsymbol{y = C_1 e^{-x} + C_2 + \frac{e^{4x}}{20}}.$$

d) Evaluate $\int_0^1 \int_{x^2}^{x} \boldsymbol{xy(x + y)dydx}$. **[3]**

ANS: We have,

$$I = \int_0^1 \int_{x^2}^{x} xy(x+y)dydx.$$

$$I = \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx$$

$$I = \int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

$$I = \left[\frac{5}{6} \cdot \frac{x^5}{5} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1$$

$$I = \frac{1}{6} - \frac{1}{14} - \frac{1}{24}$$

$$\boldsymbol{I = \frac{3}{56}}$$

e) Solve $(4x+3y-4)dx + (3x-7y-3)dy = 0$. **[4]**

ANS: Given, $(4x+3y-4)dx + (3x-7y-3)dy = 0.$

∴ M = $(4x+3y-4)$ and N = $(3x-7y-3)$

Differentiating M by y and N by x, we get,

$\frac{dM}{dy} = 3$ And $\frac{dN}{dx} = 3$

$\therefore \frac{dM}{dy} = \frac{dN}{dx}$

∴The given equations are exact.

For solution,

$\int M\,dx = \int (4x+3y-4)\,dx$

$\int M\,dx = 2x^2 + 3xy - 4x$

$\int (Term\ is\ N\ free\ from\ x) = \int -7y - 3\,dy$

$= \frac{-7y^2}{2} - 3y$

∴ The final solution is,

$$2x^2 + 3xy - 4x - \frac{7y^2}{2} - 3y = c$$

$$\boldsymbol{4x^2 + 6xy - 8x - 7y^2 - 6y = c}$$

f) Solve $\frac{dy}{dx} = 1 + xy$ with initial condition $x_0 = 0,\ y_0 = 0.2$ By Taylors series method. Find the approximate value of y for x = 0.4(step size = 0.4).

ANS: The Taylor series is given by,

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' \ldots.. \qquad \ldots\ldots.. (1)$$

With $x_0 = 0, y_0 = 0.2, x = 0.4$

Now, y' = 1 + xy $\qquad \therefore y_0' = 1$

y'' = y + xy' $\qquad \therefore y_0'' = y_0 = 0.2$

y''' = y' + y' + xy''

= 2y' + xy'' $\qquad \therefore y_0''' = 2y_0' = 2$

y'''' = 2y'' + y'' + xy''' $\quad \therefore y_0'''' = 3y_0'' + xy_0''' = 0.6$

Putting these values in equation 1, we get

$$y = 0.2 + (0.4)1 + + \frac{(0.4)^2}{2!}0.2 + \frac{(0.4)^3}{3!}.2 + \frac{(0.4)^4}{4!}.(0.6) + \cdots$$

$$y = 0.2 + 0.4 + 0.016 + 0.02133 + 0.00064$$

$$\boldsymbol{y = 0.63797.}$$

Q.2 a) Solve$\frac{d^2y}{dx^2} - 16y = x^2e^{3x} + e^{2x} - \cos 3x + 2^x$. **[6]**

ANS: The auxiliary equation is $D^2 - 16 = 0$

$\therefore D = 4, -4$

$\therefore$ The C.F. is $y = C_1e^{4x} + C_2e^{-4x}$

Now, to find P.I.,

P.I. = $\frac{1}{D^2-16}(x^2e^{3x} + e^{2x} - \cos 3x + 2^x)$

Now, $\frac{1}{D^2-16}x^2e^{3x} = e^{3x}.\frac{1}{(D+3)^2-16}.x^2$

$$= e^{3x}.\frac{1}{D^2+6D+9-16}.x^2 = e^{3x}.\frac{1}{D^2+6D-7}.x^2$$

$$= -\frac{e^{3x}}{7} . \frac{1}{(1-\frac{D^2+6D}{7})} . x^2$$

$$= -\frac{e^{3x}}{7} . (1-\frac{D^2+6D}{7})^{-1} . x^2$$

$$= -\frac{e^{3x}}{7} (1+\frac{D^2+6D}{7}+\frac{D^4+6D^3+36D^2}{49}+\cdots) . x^2$$

$$= -\frac{e^{3x}}{7} (x^2+\frac{2x+2}{1\ \ 7}+\frac{72}{49}) = -\frac{e^{3x}}{7} (x^2+\frac{2x}{17}+\frac{86}{49})$$

$$\therefore \frac{1}{D^2-16} . e^{2x} = e^{2x} \frac{1}{2^2-16} = e^{2x} . \frac{1}{2^2-16} = e^{2x} . \frac{1}{12}$$

$$\therefore \frac{1}{D^2-16} . \cos 3x = \frac{\cos 3x}{-9-16} = \frac{\cos 3x}{-25}$$

$$\therefore \frac{1}{D^2-16} . 2^x = \frac{1}{D^2-16} . e^{x\log 2} = \frac{(\ e^{x\log 2}}{\log 2)^2-16} . = \frac{2^x}{(\log 2)^2-16}.$$

$\therefore$P.I. = $-\frac{e^{3x}}{7}(x^2+\frac{12x}{7}+\frac{86}{49}) + e^{2x} . \frac{1}{12} + \frac{\cos 3x}{25} + \frac{2^x}{(\log 2)^2-16}.$

$\therefore$ The complete equation is,

$$\boldsymbol{y = C_1 e^{4x} + C_2 e^{-4x} - \frac{e^{3x}}{7}(x^2+\frac{12x}{7}+\frac{86}{49}) + e^{2x} . \frac{1}{12} + \frac{\cos 3x}{25} + \frac{2^x}{(\log 2)^2-16}}.$$

b) Show that $\boldsymbol{\int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x} dx = \pi \sin^{-1} a \ \ 0 \le a \le 1}$. **[6]**

ANS: Let I (a) be the given integral. By the rule of differentiation under the integral sign.

$$\frac{dI}{da} = \int_0^{\pi} \frac{df}{da}\, dx = \int_0^{\pi} \frac{1}{\cos x} . \frac{\cos x}{1+a\cos x}\, dx = \int_0^{\pi} \frac{dx}{1+a\cos x}$$

Put $t = tan\frac{x}{2}, dx = \frac{2dt}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

When x = 0, t = 0;

When x =π, t = $tan\frac{\pi}{2}$=∞

$$\therefore \frac{dI}{da} = \int_0^\infty \frac{1}{1+a.(\frac{1-t^2}{1+t^2})} \cdot \frac{2\,dt}{1+t^2}$$

$$\frac{dI}{da} = \int_0^\infty \frac{2\,dt}{(1+t^2)+a(1-t^2)} .$$

$$\frac{dI}{da} = \int_0^\infty \frac{2\,dt}{(1+a)+(1-a)t^2} .$$

$$\frac{dI}{da} = \frac{1}{1-a} \int_0^\infty \frac{2\,dt}{[\frac{1+a}{1-a}]+t^2}$$

$$\frac{dI}{da} = \frac{2}{1-a} \sqrt{\frac{1-a}{1+a}} . \left[\tan^{-1} \sqrt{\frac{1-a}{1+a}}\right]_0^\infty$$

$$\frac{dI}{da} = \frac{2}{\sqrt{1-a^2}} . \frac{\pi}{2}$$

$$\frac{dI}{da} = \frac{\pi}{\sqrt{1-a^2}} .$$

Integrating both sides w.r.t. a, we get

$$I = \pi \ \sin^{-1} a + c$$

To find c, put a = 0

$$I(0) = \pi \ \sin^{-1} 0 + c, c = 0$$

$$\therefore I = \pi \sin^{-1} a$$

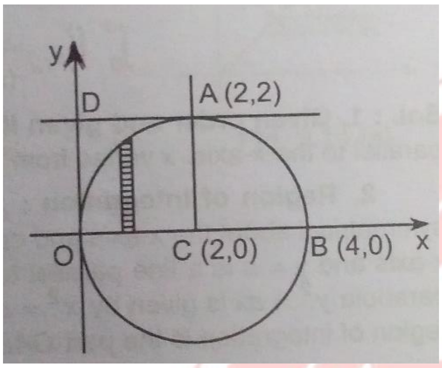
$$\boxed{\therefore \int_0^\pi \frac{\log(1+a\cos x)}{\cos x} dx = \pi \sin^{-1} a}$$

c) Change the order of integration and evaluate $\int_0^2 \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dxdy$. [8]

ANS: 1) Given order and given limits: Given order is: first w.r.t. x and then w.r.t y i.e., a strip parallel to the x-axis varies from x = 2 – $\sqrt{4-y^2}$ to x = 2 – $\sqrt{4-y^2}$. Y varies from y = 0 to y = 2.

2) Region of integration: x = 2 – $\sqrt{4-y^2}$ is the arc and x = 2 + $\sqrt{4-y^2}$ is the arc of the circle $(x - 2)^2 + y^2 = 4$with centre at (2, 0) and radius = 2 above the x-axis. y = 0 is the x-axis and y = 2 is the line parallel to the x-axis through A (2, 2). The region of integration is the semi-circle OAB above the x-axis. The points of intersection of the circle and the x-axis are O (0, 0) and B (4, 0).

3) Change of order of integration: To change the order, consider a strip parallel to the y-axis in the region of integration. On this strip y varies from y = 0 to y =$\sqrt{4-(x-2)^2}$ and then strip moves from x = 0 to x = 4.



$$I = \int_0^4 \int_0^{\sqrt{4-(x-2)^2}} dydx$$

$$I = \int_0^4 [y]_0^{\sqrt{4-(x-2)^2}} dx$$

$$I = \int_0^4 \sqrt{4-(x-2)^2}\, dx$$

$$I = \left[\frac{x-2}{2}\sqrt{4-(x-2)^2} + 2\sin^{-1}\frac{x-2}{2}\right]_0^4$$

$$I = (2.\frac{\pi}{2}) - (-2.\frac{\pi}{2})$$

$$\therefore \boldsymbol{I = 2\pi}$$

Q.3 a) Evaluate $\iiint (x+y+z)\, dxdydz$ over the tetrahedron bounded by the planes x = 0, y = 0, z = 0 and x + y + z = 1. [6]

ANS:

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} (x+y+z) dzdydx$$

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \left[\frac{(x+y+z)^2}{2}\right]_0^{1-x-y} dydz$$

$$I = \frac{1}{2}\int_{x=0}^{1} \int_{y=0}^{1-x} [1-(x+y)^2] dydx$$

$$I = \frac{1}{2}\int_{x=0}^{1} \left[y - \frac{(x+y)^2}{2}\right]_0^{1-x} dx$$

$$I = \frac{1}{2}\int_{x=0}^{1}\left[(1-x) - \frac{1}{3} + \frac{x^3}{3}\right]dx$$

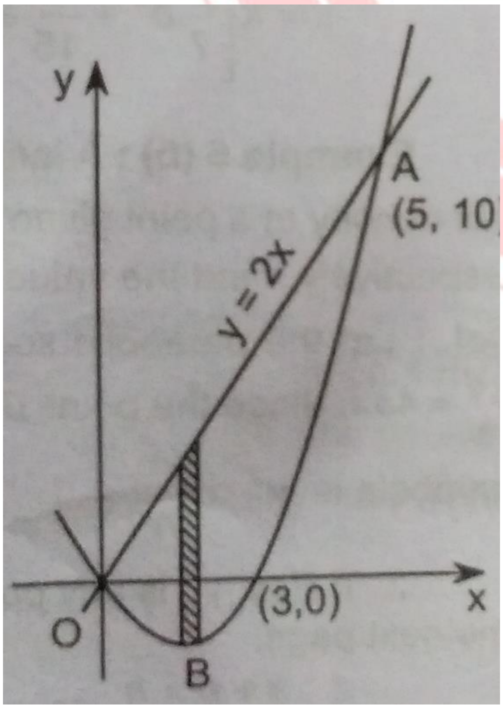
$$I = \frac{1}{2}\left[\frac{2x}{3} - \frac{x^2}{2} + \frac{x^4}{12}\right]_0^1$$

$$I = \frac{1}{2}.\frac{3}{12} = \frac{1}{8}$$

$$\therefore \boldsymbol{I = \frac{1}{8}}$$

b) Find the mass of lamina bounded by the curves $y = x^2 - 3x$ and $y = 2x$ if the density of the lamina at any point is given by $\frac{24}{25}xy$. [6]

ANS: The curve $y = x^2 - 3x$ i.e. $y + \frac{9}{4} = (x - \frac{3}{2})^2$ is parabola intersecting the x-axis in x = 0 and x = 3. The line y = 2x intersects this parabola at $x^2 - 3x = 2x$ i.e. $x^2 - 5x = 0$ i.e. at x = 0, x = 5. Therefore, points of intersection are (0,0) and (5,10). The surface density is ρ = (24/25)xy. Taking the elementary strip parallel to the y-axis, on the strip y varies from $y = x^2 - 3x$ to $y = 2x$ and then x varies from x = 0 to x = 5.



∴ Mass of lamina = $_{0} \quad _{x^2-3x} \, _{25} \, xydxdy$

$$= ^{24} \quad ^{5} x \left[^{y^2}_{2}\right]^{2x}_{x^2-3x} dx$$

$$= ^{24} \int^{5} 4x^3 - x(x^4 - 6x^3 + 9x^2)]dx$$

$$= ^{24} \int^{5} -5x^3 + 6x^4 - x^5)]dx$$

$$\frac{24}{50} \quad \frac{-x^6}{b} \quad \frac{6x^5}{5} \quad \frac{5x^4}{4} \quad {}^{5}_{0}$$

$$= ^{24} . 5^4[- ^{25} + 6 - ^{5}]$$

$$= _{50} . 5^4 . _{12}$$

∴**Mass of lamina = 175.**

c) Solve $\boldsymbol{x^2 \, ^{dxy} + 3x \, ^{dy} + 3y = \, ^{\log x \, .\cos(\log x)}}$ **[8]**

ANS: Given that,

$$x^2 \, {}^{d \;\, y}_{\;\; x} + 3x \, ^{dy} + 3y = \, ^{\log x \, .\cos(\log x)}$$

Putting z = log x and x = e^z, we get

$$[D(D-1) + 3D + 3]y = e^{-z}. z. \cos z$$

$$[D^2 + 2D + 3]y = e^{-z}. z. \cos z$$

∴The A.E. is $D^2 + 2D + 3 = 0$

$$\therefore D = {}^{-2 \quad 2\sqrt{2}.i}_{\pm} = -1 \pm \sqrt{2}. i$$

∴The C.F. is $y = e^{-z}(C \;\; \cos\sqrt{2}z + C_2 \sin\sqrt{2}z)$

P.I. = $_{D^2+2D+3} \, e^{-z}. z. \cos z$

$$= e^{-z}. \, {}^{\qquad 2}_{(D-1)^2 + \;\; (D-1)+3} . z. \cos z = e^{-z}. \, _{D^2+2} . z. \cos z$$

$= e^{-z}\,[z - \frac{}{D^2+2}\,.\,2D]\,.\,\frac{}{D^2+2}\,.\cos z$

$= e^{-z}\,[z - \frac{}{D^2+2}\,.\,2D]\cos z = e^{-z}\,[z cos\, z + \frac{}{D^2+2}\,.\,2\sin z]$

$= e^{-z}[z\cos z + 2\sin z]$

The complete solution is,

y = C.F. + P.I.

$y = e^{-z}(C \;\; \cos\sqrt{2}z + C_2\sin\sqrt{2}z) + e^{-z}[z\cos z + 2\sin z]$

$\boldsymbol{y = {}^{1}\,(C_1\cos\sqrt{2}\log x + C_2\sin\sqrt{2}\log x) + {}^{1}\,[\log x\cos\log x}$

$\boldsymbol{+2\sin\log x]}$

Q.4 a) Find by double integration the area bounded by the parabola

$\boldsymbol{y^2 = 4x}$ **And** $\boldsymbol{y = 2x - 4}$ **[6]**

ANS: The parabola $y^2 = 4x$ and the line $y = 2x - 4$ intersect where $(2x - 4)^2 = 4x$

$\therefore 4x^2 - 16x + 16 = 4x$ $\therefore 4x^2 - 20x + 16 = 0$

$\therefore x^2 - 5x + 4 = 0$ $\therefore$(x - 4) (x - 1)=0

$\therefore$x = 1, 4.

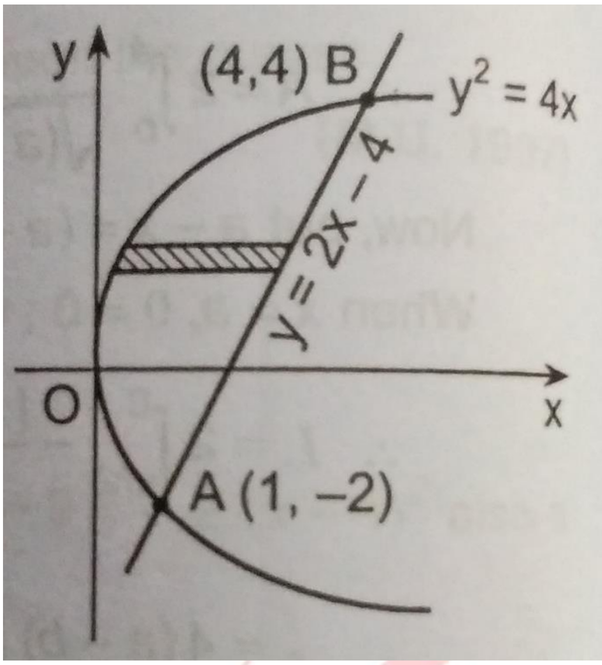
When x = 1, y = 2 – 4 = -2; and when x = 4, y = 8 - 4 = 4. Thus, the points of intersection are A (1, -2) and B (4, 4).

Now, consider a strip parallel to x-axis. On this strip x varies from x = y^2/4 to x = (y+4)/2. The strip then moves parallel to the x-axis from y = -2 to y = 4.

$$\int_{-2}^{4}\int_{y^2/4}^{(y+4)/2} \qquad \int_{-2}^{4} \frac{y+4}{} \;\; \frac{y^2}{4}$$

$$= \int_{-2}(\frac{y+4}{} - \frac{y^2}{})dy$$

$$= \frac{1}{}\int_{-2}(2y + 8 - y^2)dy$$



$$= \frac{1}{4}\left[y^2 + 8y - \frac{y^3}{3}\right]_{-2}^{4}$$

$$= \frac{1}{4}[(16 + 32 - \frac{64}{3}) - (4 - 16 + \frac{8}{3})]$$

$$= \frac{1}{4}(60 - 24)$$

$\therefore \boldsymbol{A = 9}$

b) Solve $\frac{dy}{dx} + x\sin 2y = x^3cos^2y$ [6]

ANS: Given, $\frac{dy}{dx} + x\sin 2y = x^3cos^2y$

Dividing both sides by $\cos^2x$,

$$sec^2x\,\frac{dy}{dx} + x\;sec^2x\sin 2y = x^3$$

$$sec^2x\,\frac{dy}{dx} + 2x\tan y\; = x^3 \text{……………………. (1)}$$

Put tan y = v and differentiate w.r.t. x,

$$sec^2x\,\frac{dy}{dx} = \frac{dv}{dx}$$

Hence, from (1), we get $\frac{dv}{dx} + 2v.x = x^3$

$\therefore\ P = 2x$ And Q = x^3

$\therefore \int P\,dx = \int 2x\,dx = x^2$

$\therefore$I.F. = $e^{\int P\,dx} = e^{\int 2x\,dx} = e^{x^2}$

$\therefore$The solution is $v\,e^{x^2} = \int e^{x^2} x^3\,dx + c$

To find the integral put $x^2 = t,\ xdx = \frac{dt}{2}$.

$\therefore I = \int e^t . t . \frac{dt}{2} = \frac{1}{2}[te^t - \int e^t . dt]$……….. [By parts]

$\therefore I = \frac{1}{2}[te^t - e^t] = \frac{1}{2}e^t(t-1) = \frac{1}{2}e^{x^2}(x^2-1)$

$\therefore$ The solution is $v\,e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c$

$\therefore \boldsymbol{\tan y\, e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c}$

c) Solve $\frac{dy}{dx} = x^3 + y$ with initial conditions $y(0) = 2$ at $x = 0.2$ in step of

h = 0.1 by Runge Kutta method of Fourth order. [8]

ANS: Given that, $\frac{dy}{dx} = x^3 + y$

$f(x,y) = x^3 + y, x_0 = 0, y_0 = 2\ and\ h = 0.1$

$\therefore k_1 = hf(x_0, y_0) = 0.1(0+2) = 0.2$

$\therefore k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1[(\frac{0.1}{2})^3 + 2 + \frac{0.2}{2}] = 0.2100$

$\therefore k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1[(\frac{0.1}{2})^3 + 2 + \frac{0.2100}{2}] = 0.2105$

$\therefore k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_3}{2}\right) = 0.1[(\frac{0.1}{2})^3 + 2 + \frac{0.2100}{2}] = 0.23105$

$\therefore k = \frac{k_1+2k_2+2k_3+k_4}{6} = \frac{0.2+2(0.21)+2(0\ 2105)+0.23105}{6}$

$\therefore \boldsymbol{k = 0.2120}$

Q.5 a) Evaluate $\int_0^1 x^5 \sin^{-1} x\, dx$ and find the value of $\beta(\frac{9}{2}, \frac{1}{2})$. **[6]**

ANS: $I = \int_0^1 x^5 \sin^{-1} x\, dx$

Put $\sin^{-1} x = t$ $\therefore$ x = sin t $dx = \cos t\, dt$

When x = 0, t = 0 when x = 1, t = $\pi/2$

$$I = \int_0^{\pi/2} \sin^5 t . t . \cos t\, dt = \int_0^{\pi/2} t(\sin^5 t . \cos t)\, dt$$

Integrating by parts,

$$I = \left[t . \frac{sin^6 x}{6}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{sin^6 x}{6} . 1 . dt$$

$$I = \left(\frac{\pi}{2} . \frac{1}{6} - 0\right) - \frac{1}{6} . \frac{5.3.1}{6.4.2} . \frac{\pi}{2}$$

$$I = \frac{\pi}{12} - \frac{5\pi}{192}$$

$$\therefore \boldsymbol{I} = \frac{\mathbf{11\pi}}{\mathbf{192}}$$

$$\beta\left(\frac{9}{2}, \frac{1}{2}\right) = \frac{\left|\frac{9}{2}\right|\frac{1}{2}}{\left|5\right.} = \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\left|\frac{1}{2}\right|\frac{1}{2}}{5.4.3.2.1}$$

$$\beta\left(\frac{9}{2}, \frac{1}{2}\right) = \frac{105\pi}{384}$$

b) In a circuit containing inductance L, resistance R, and voltage E, the current i is given by $L\frac{di}{dt} + Ri = E$. Find the current i at time t at t = 0 and i = 0 and L, R and E are constants. **[6]**

ANS: The given equation $\frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$ is linear of the type $\frac{dy}{dx} + Py = Q$

$\therefore$ Its solution is $ie^{\int R/_L dt} = \int e^{\int R/_L dt} . \frac{E}{L} . dt + c$

$$i . e^{Rt/_L} = \frac{E}{L}\int e^{Rt/_L}\ dt + c = \frac{E}{L} . e^{Rt/_L} \frac{L}{R} + c$$

$$= \frac{E}{R} e^{Rt/_L} + c$$

When t = 0 and i = 0 $\therefore c = -\frac{E}{R}$

$\therefore i.e^{Rt/L} = \frac{E}{R}e^{Rt/L} - \frac{E}{R}$

$\therefore i = \frac{E}{R}(e^{Rt/L} - 1)$

$\therefore \boldsymbol{i = \frac{E}{R}(1 - e^{-Rt/L})}$

c) Evaluate $\int_0^6 \frac{dx}{1+3x}$ by using 1} Trapezoidal 2} Simpsons (1/3) rd. and 3} Simpsons (3/8) Th rule. [8]

ANS:

X	0	1	2	3	4	5	6
Y	1	0.25	0.1428	0.1	0.0769	0.0625	0.0526
Ordinate	y_0	y_1	y_2	y_3	y_4	y_5	y_6

1} Trapezoidal Rule:

I = $\frac{h}{2}(X + 2R)$

X = Sum of extreme value = 1 + 0.0526 = 1.0526

R = Sum of Remaining values = 0.25 + 0.1428 + 0.1 + 0.0769 + 0.0625 = 0.6322

I = $\frac{1}{2}(1.0526 + 2(0.6322))$

$\boldsymbol{I = 1.1585}$

2} Simpsons (1/3) rd rule

$I = \frac{h}{3}(X + 2E + 4O)$

X = Sum of Extreme values =1 + 0.0526 = 1.0526

E = Sum of even ordinates = 0.1428 + 0.0769 = 0.2197

O = Sum of odd ordinates = 0.25 + 0.1 + 0.0625 = 0.4125

$I = \frac{1}{3}(1.0526 + 2(0.2197) + 4(0.4125))$

$\boldsymbol{I = 0.5616}$.

3} Simpsons (3/8) Th rule.

$I = \frac{3h}{8}(X + 2T + 4R)$

X = Sum of extreme value = 1 + 0.0526 = 1.0526

T = Sum of multiple of three =0.1

R = Sum of Remaining values =0.25 + 0.1428 +0.0769 + 0.0625 = 0.5322

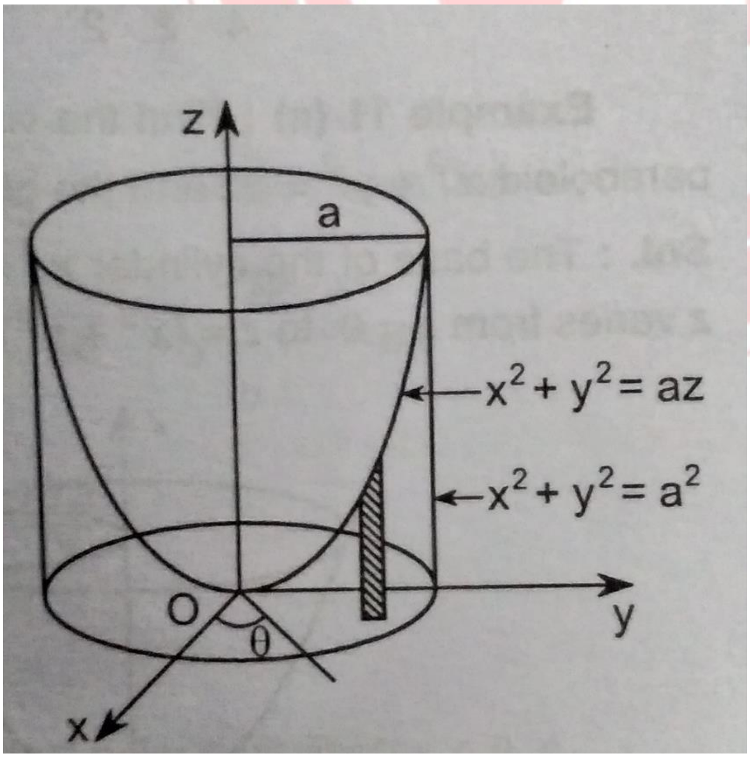
$I = \frac{3*1}{8}(1.0526 + 2(0.1) + 4(0.5322))$

$\boldsymbol{I = 1.06845}$.

Q.6 a) Find the volume bounded by the paraboloid $x^2 + y^2 = az$ and the cylinder $x^2 + y^2 = a^2$. [6]

ANS: The equations of the cylinder and the paraboloid in polar form are r = a and r^2 = az.

Now, z varies from z = 0 to z = r^2/a, r varies from r = 0 to r = a and θ varies from θ = 0 to θ = $\frac{\pi}{2}$ taken 4 times.



$$\therefore V = 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a}\int_{z=0}^{r^2/a} r\,dr\,d\theta\,dz$$

$$\therefore V = 4\int_{\theta=0}^{\frac{\pi}{2}} r\,[z]_0^{r^2/a} dr d\theta$$

$$\therefore V = 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a}\frac{r^3}{a}dr\,d\theta$$

$$\therefore V = \frac{4}{a}\int_{\theta=0}^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^a d\theta$$

$$\therefore V = \frac{4}{a}\int_0^{\frac{\pi}{2}}\frac{a^4}{4}\,d\theta$$

$$\therefore V = a^3\int_0^{\frac{\pi}{2}} d\theta$$

$$\therefore V = a^3[\theta]_0^{\pi/2}$$

$$\therefore \boldsymbol{V} = \frac{\boldsymbol{\pi a^3}}{\boldsymbol{2}}$$

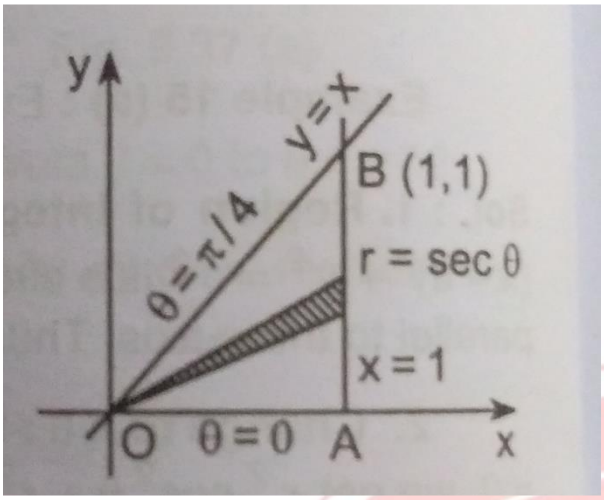
b) Change to polar coordinates and evaluate $\int_0^1\int_0^x (x+y)\,dy\,dx$**.**
[6]

ANS: 1) Region of integration: y= 0 is the x-axis and y = x is a line OB through the origin; x = 0 is the y-axis and x = 1 is a line AB parallel to the y-axis. Thus the region of integration is the triangle OAB.

2) Change to r, θ: Putting x = r cos θ and y = r sin θ, the line y = x becomes r sin θ= r cos θ i.e. tan θ = 1 i.e. θ = $\frac{\pi}{4}$. The x-axis is given by θ = 0 and the y-axis is given by θ =$\frac{\pi}{2}$. And line x = 1 is given by r cos θ = 1 i.e. r = sec θ.

3) Integrand: Putting x = r cos θ and y = r sin θ in (x + y), we get,

r cos θ + r sin θ = r(cos θ + sin θ) and dydx is replaced by r drdθ



$$\therefore I = \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \mathrm{r}(\cos\theta + \sin\theta)\mathrm{r}dr\, d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} (\cos\theta + \sin\theta)\mathrm{r}^2 dr\, d\theta$$

$$I = \int_0^{\frac{\pi}{4}} (\cos\theta + \sin\theta)\left[\frac{r^3}{3}\right]_0^{\sec\theta} d\theta$$

$$I = \frac{1}{3}\int_0^{\frac{\pi}{4}} (\cos\theta + \sin\theta)\, sec^3\theta\, d\theta$$

$$I = \frac{1}{3}\left[\int_0^{\frac{\pi}{4}} sec^2\theta\, d\theta + \int_0^{\frac{\pi}{4}} \frac{1}{cos^3\theta}\sin\theta\, d\theta\right]$$

$$I = \frac{1}{3}\left[\tan\theta + \frac{1}{2cos^2\theta}\right]_0^{\frac{\pi}{4}}$$

$$I = \frac{1}{3}(1 + \frac{1}{2}(2-1))$$

$$\boldsymbol{I = \frac{1}{2}}$$

c) Solve by method of variation of parameters

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}. \qquad \textbf{[8]}$$

ANS: A.E: $D^2 + 3D + 2 = 0$

Solving the equation, we get

$\therefore$ D = -1, -2.

$\therefore$ C.F = C1e^{-x} + C2e^{-2x}.

$\therefore$ y_1=e^{-x} y_2= e^{-2x}

$\therefore$ $y_1' = -e^{-x}$ $y_2' = -2e^{-2x}$

$\therefore w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$

$= -2e^{-3x} + e^{-3x}$

$= -e^{-3x}$

X =e^{e^x}.

$\therefore u = -\int \frac{y_2 X}{w}\,dx$

$= -\int \frac{e^{-2x}e^{e^x}}{-e^{-3x}}\,dx$

$= -\int e^{e^x}.e^x\,dx$

Put e^x = t

e^x dx =dt

$\therefore \int e^t dt = e^t + c.$

$\therefore w = e^{e^x} + c$

$v = \int \frac{y_1 X}{w}\,dx$

$v = \int \frac{e^{-x}\,e^{e^x}}{-e^{-3x}}\,dx$

$v = \int e^{e^x} e^{2x}\,dx$

Putting e^x = t

$\therefore v = \int e^t.t\,dt = te^t - e^t$

$\therefore v = e^x e^{e^x} - e^{e^x}$

$\therefore$ P.I. = $uy_1 + vy_2 = e^{e^x}.e^{-x} - (e^x e^{e^x} - e^{e^x})\, e^{-2x}$

$= e^{-2x}.\ e^{e^x}$

$\therefore$ The complete solution is,

y = C.F + P.I

$y = C_1 e^{-x} + C_2 e^{-2x} + e^{-2x}.\ e^{e^x}$